Lecture 1

Chapter 1

Linear equations

SECTION 1.3

Graphs of linear equations

Objectives

At the end of this section you should be able to:

- Plot points on graph paper given their coordinates.
- Sketch a line by finding the coordinates of two points on the line.
- Solve simultaneous linear equations graphically.
- Sketch a line by using its slope and intercept.

Plot points on the graph paper given their coordinates

Consider the two straight lines shown in Figure 1.1. The horizontal line is referred to as the **x axis** and the vertical line is referred to as the **y axis**. The point where these lines intersect is known as the **origin** and is denoted by the letter O. These lines enable us to identify uniquely any point, P, in terms of its **coordinates** (x, y). The first number, x, denotes the horizontal distance along the x axis and the second number, y, denotes the vertical distance along the y axis. The arrows on the axes indicate the positive direction in each case.

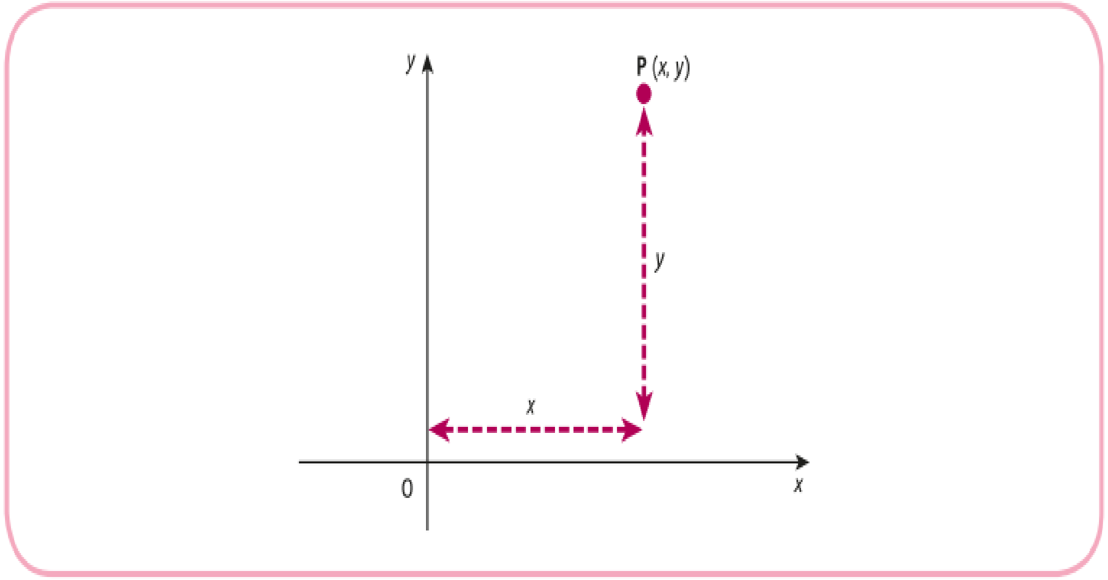

Figure 1.1

Figure 1.2 shows the five points A(2, 3), B(−1, 4), C(−3, −1), D(3, −2) and E(5, 0) plotted on coordinate axes. The point A with coordinates (2, 3) is obtained by starting at the origin, moving 2 units to the right and then moving 3 units vertically upwards. Similarly, the point B with coordinates (−1, 4) is located 1 unit to the left of O (because the x coordinate is negative) and 4 units up.

Note that the point C lies in the bottom left-hand quadrant since its x and y coordinates are both negative. It is also worth noticing that E actually lies on the x axis since its y coordinate is zero. Likewise, a point with coordinates of the form $(0, y)$ for some number y would lie somewhere on the y axis. Of course, the point with coordinates (0, 0) is the origin, O.

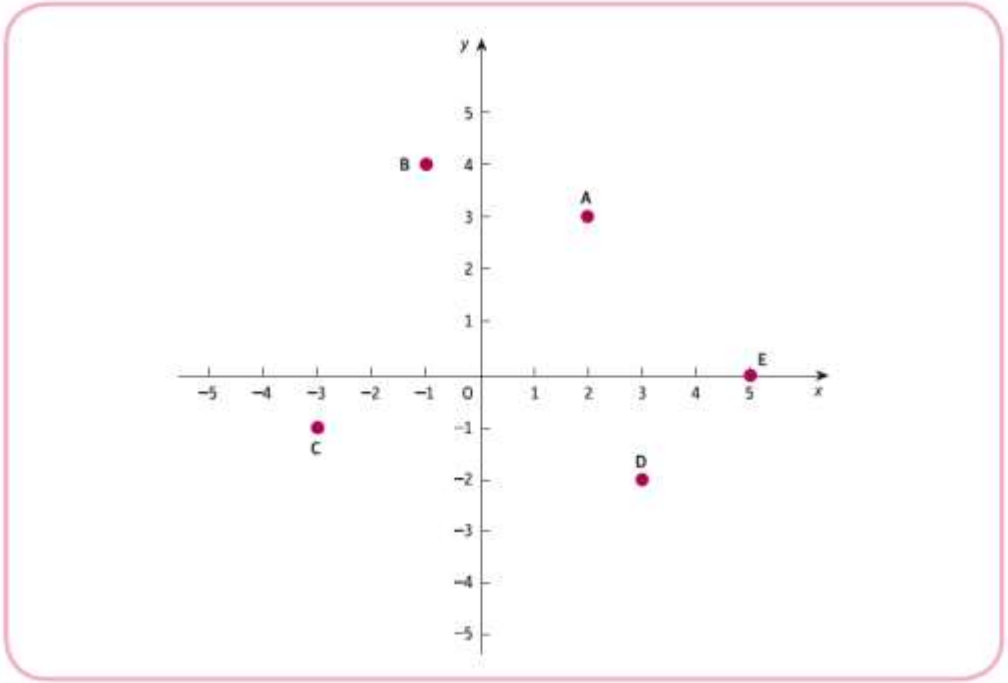

Figure 1.2

Practice Problem

1. Plot the following points on graph paper. What do you observe?

(2, 5), (1, 3), (0, 1), (−2, −3), (−3, −5)

In economics we need to do rather more than just plot individual points on graph paper. We would like to be able to sketch curves represented by equations and to deduce information from such a picture. We restrict our attention in this section to those equations whose graphs are straight lines, deferring consideration of more general curve sketching until Chapter 2.

In Practice Problem 1 you will have noticed that the five points (2, 5), (1, 3), (0, 1), (−2, −3) and (−3, −5) all lie on a straight line. In fact, the equation of this line is

$$-2x + y = 1$$

Any point lies on this line if its x and y coordinates satisfy this equation. For example, (2, 5) lies on the line because when the values $x = 2$, $y = 5$ are substituted into the left-hand side of the equation, we obtain

$$-2(2) + 5 = -4 + 5 = 1$$

which is the right-hand side of the equation. The other points can be checked similarly (see Table 1.1).

Table 1.1

Point	Check	
$(1, 3)$	$-2(1) + 3 = -2 + 3 = 1$	✓
$(0, 1)$	$-2(0) + 1 = 0 + 1 = 1$	✓
$(-2, -3)$	$-2(-2) - 3 = 4 - 3 = 1$	✓
$(-3, -5)$	$-2(-3) - 5 = 6 - 5 = 1$	✓

Sketch a line by finding the coordinates of two points on the line

The general equation of a straight line takes the form

a multiple of x + a multiple of y = a number

that is,

$$dx + ey = f$$

for some given numbers d, e and f. Consequently, such an equation is called a **linear equation**. The numbers d and e are referred to as the **coefficients**. The coefficients of the linear equation,

$$-2x + y = 1$$

are -2 and 1 (the coefficient of y is 1 because y can be thought of as $1 \times y$).

Practice Problem

2. Check that the points

$$(-1, 2), (-4, 4), (5, -2), (2, 0)$$

all lie on the line

$$2x + 3y = 4$$

and hence sketch this line on graph paper. Does the point $(3, -1)$ lie on this line?

In general, to sketch a line from its mathematical equation, it is sufficient to calculate the coordinates of any two distinct points lying on it. These two points can be plotted on graph paper and a ruler used to draw the line passing through them. One way of finding the coordinates of a point on a line is simply to choose a numerical value for x and to substitute it into the equation. The equation can then be used to deduce the corresponding value of y. The whole process can be repeated to find the coordinates of the second point by choosing another value for x.

Example

Sketch the line

$$4x + 3y = 11$$

Solution

For the first point, let us choose $x = 5$. Substitution of this number into the equation gives

$$4(5) + 3y = 11$$
$$20 + 3y = 11$$

The problem now is to solve this equation for y:

$$3y = -9 \quad \text{(subtract 20 from both sides)}$$
$$y = -3 \quad \text{(divide both sides by 3)}$$

Consequently, the coordinates of one point on the line are $(5, -3)$.

For the second point, let us choose $x = -1$. Substitution of this number into the equation gives

$$4(-1) + 3y = 11$$

$$-4 + 3y = 11$$

This can be solved for y as follows:

$$3y = 15 \quad \text{(add 4 to both sides)}$$

$$y = 5 \quad \text{(divide both sides by 3)}$$

Hence $(-1, 5)$ lies on the line, which can now be sketched on graph paper as shown in Figure 1.3.

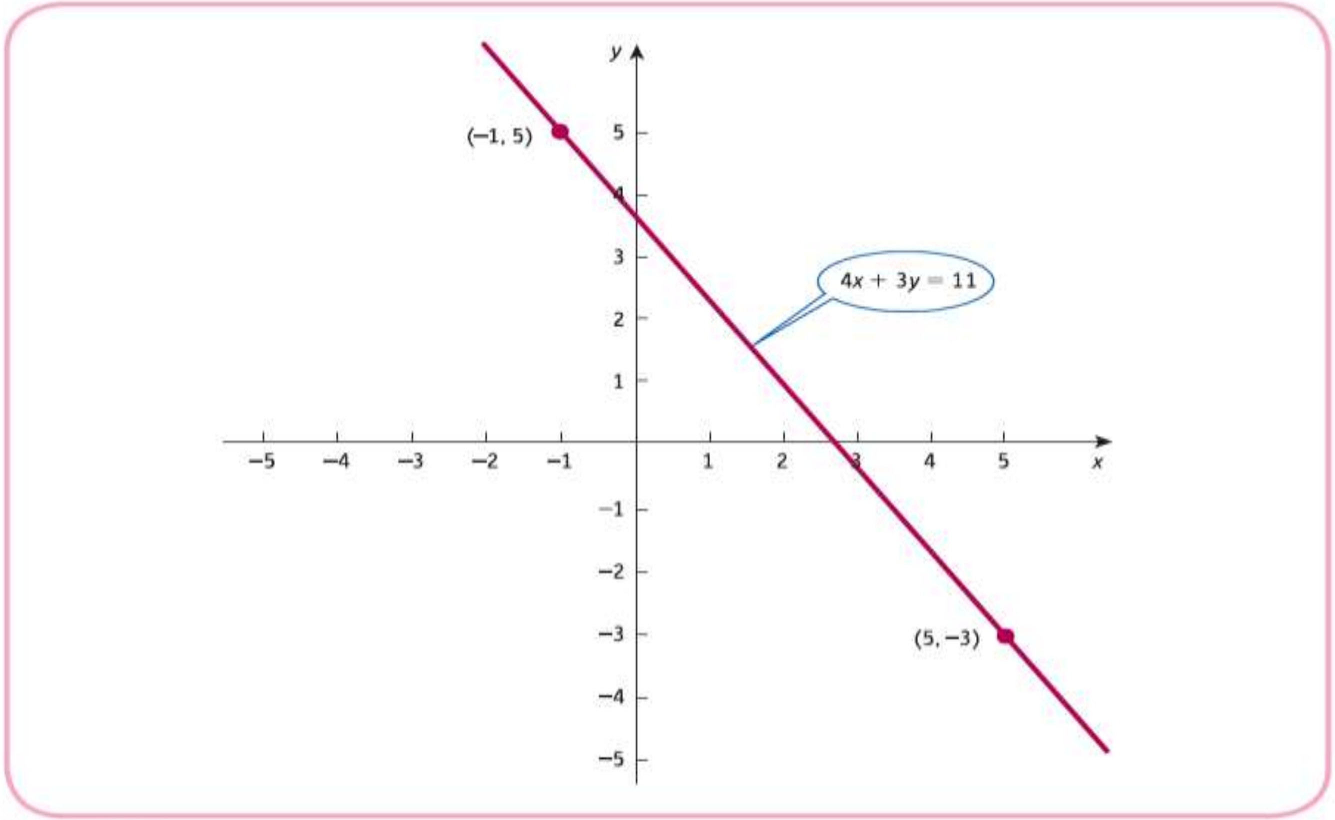

Figure 1.3

Practice Problem

3. Find the coordinates of two points on the line

$$3x - 2y = 4$$

by taking $x = 2$ for the first point and $x = -2$ for the second point. Hence sketch its graph.

Example

Sketch the line

$$2x + y = 5$$

Solution

Setting $x = 0$ gives

$$2(0) + y = 5$$
$$0 + y = 5$$
$$y = 5$$

Hence (0, 5) lies on the line.

Setting $y = 0$ gives

$$2x + 0 = 5$$
$$2x = 5$$
$$x = 5/2 \qquad \text{(divide both sides by 2)}$$

Hence (5/2, 0) lies on the line.

The line $2x + y = 5$ is sketched in Figure 1.4. Notice how easy the algebra is using this approach. The two points themselves are also slightly more meaningful. They are the points where the line intersects the coordinate axes.

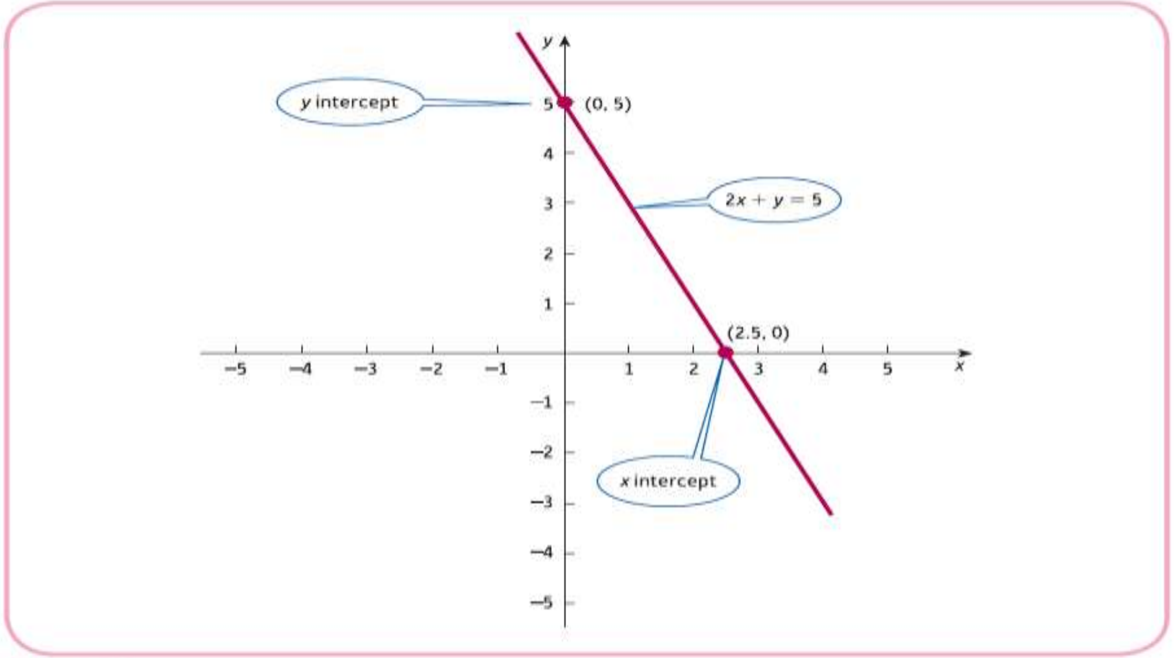

Figure 1.4

Practice Problem

4. Find the coordinates of the points where the line

$$x - 2y = 2$$

intersects the axes. Hence sketch its graph.

Example

Find the point of intersection of the two lines

$$4x + 3y = 11$$
$$2x + y = 5$$

Solution

We have already seen how to sketch these lines in the previous two examples. We discovered that

$$4x + 3y = 11$$

passes through $(5, -3)$ and $(-1, 5)$, and that

$$2x + y = 5$$

passes through $(0, 5)$ and $(5/2, 0)$.

These two lines are sketched on the same diagram in Figure 1.5, from which the point of intersection is seen to be $(2, 1)$.

It is easy to verify that we have not made any mistakes by checking that $(2, 1)$ lies on both lines.

It lies on $4x + 3y = 11$ because $4(2) + 3(1) = 8 + 3 = 11$ ✓

and lies on $2x + y = 5$ because $2(2) + 1 = 4 + 1 = 5$. ✓

For this reason, we say that $x = 2$, $y = 1$ is the solution of the **simultaneous linear equations**

$$4x + 3y = 11$$
$$2x + y = 5$$

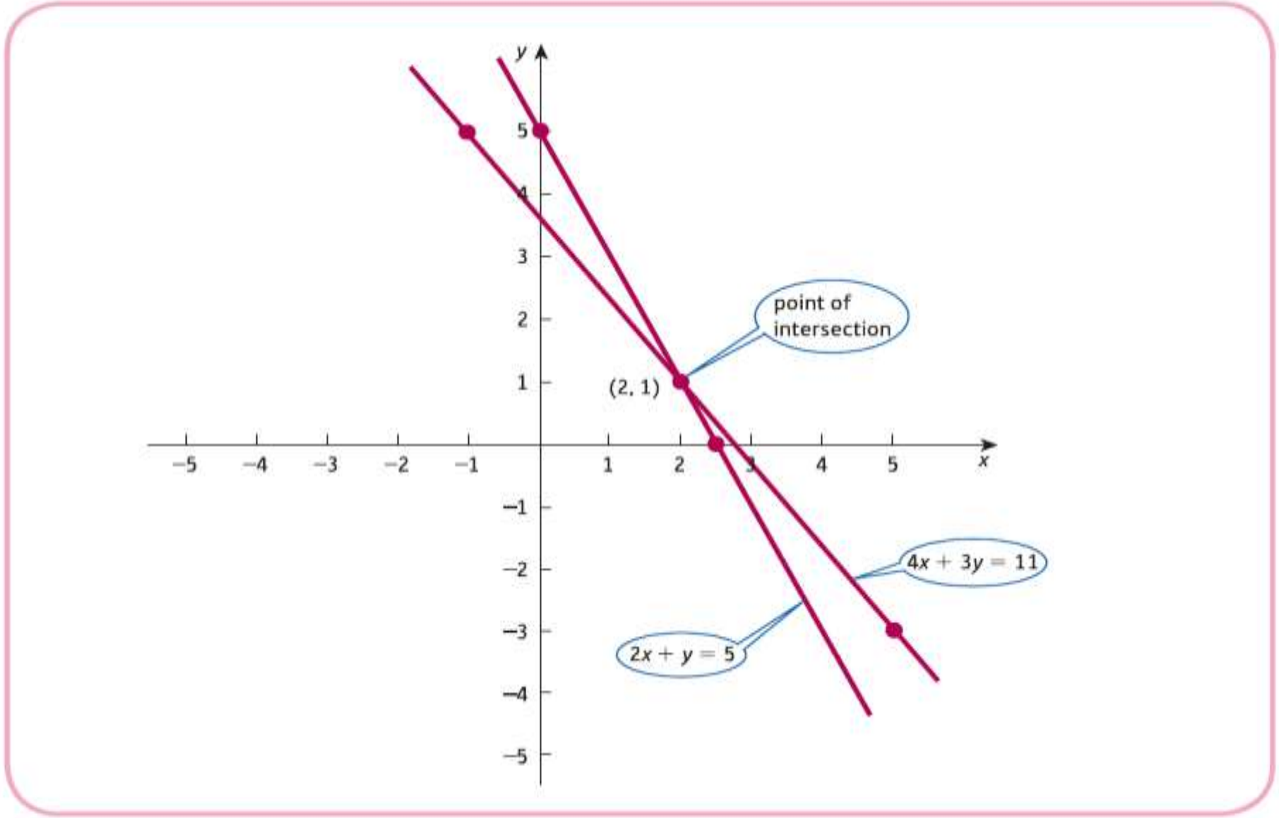

Figure 1.5

Practice Problem

5. Find the point of intersection of

$$3x - 2y = 4$$
$$x - 2y = 2$$

[Hint: you might find your answers to Problems 3 and 4 useful.]

Sketch a line by using its slope and intercept

Quite often it is not necessary to produce an accurate plot of an equation. All that may be required is an indication of the general shape together with a few key points or features. It can be shown that, provided e is non-zero, any equation given by

$$dx + ey = f$$

can be rearranged into the special form

$$y = ax + b$$

An example showing you how to perform such a rearrangement will be considered in a moment. The coefficients a and b have particular significance, which we now examine. To be specific, consider

$$y = 2x - 3$$

in which $a = 2$ and $b = -3$.

When x is taken to be zero, the value of y is

$$y = 2(0) - 3 = -3$$

The line passes through $(0, -3)$, so the y intercept is -3. This is just the value of b. In other words, the constant term, b, represents the **intercept** on the y axis.

In the same way it is easy to see that a, the coefficient of x, determines the **slope** of a line. The slope of a straight line is simply the change in the value of y brought about by a 1 unit increase in the value of x. For the equation

$$y = 2x - 3$$

let us choose $x = 5$ and increase this by a single unit to get $x = 6$. The corresponding values of y are then, respectively,

$$y = 2(5) - 3 = 10 - 3 = 7$$

$$y = 2(6) - 3 = 12 - 3 = 9$$

The value of y increases by 2 units when x rises by 1 unit. The slope of the line is therefore 2, which is the value of a. The slope of a line is fixed throughout its length, so it is immaterial which two points are taken. The particular choice of $x = 5$ and $x = 6$ was entirely arbitrary. You might like to convince yourself of this by choosing two other points, such as $x = 20$ and $x = 21$, and repeating the previous calculations.

A graph of the line

$$y = 2x - 3$$

is sketched in Figure 1.6. This is sketched using the information that the intercept is -3 and that for every 1 unit along we go 2 units up. In this example the coefficient of x is positive. This does not have to be the case. If a is negative, then for every increase in x there is a

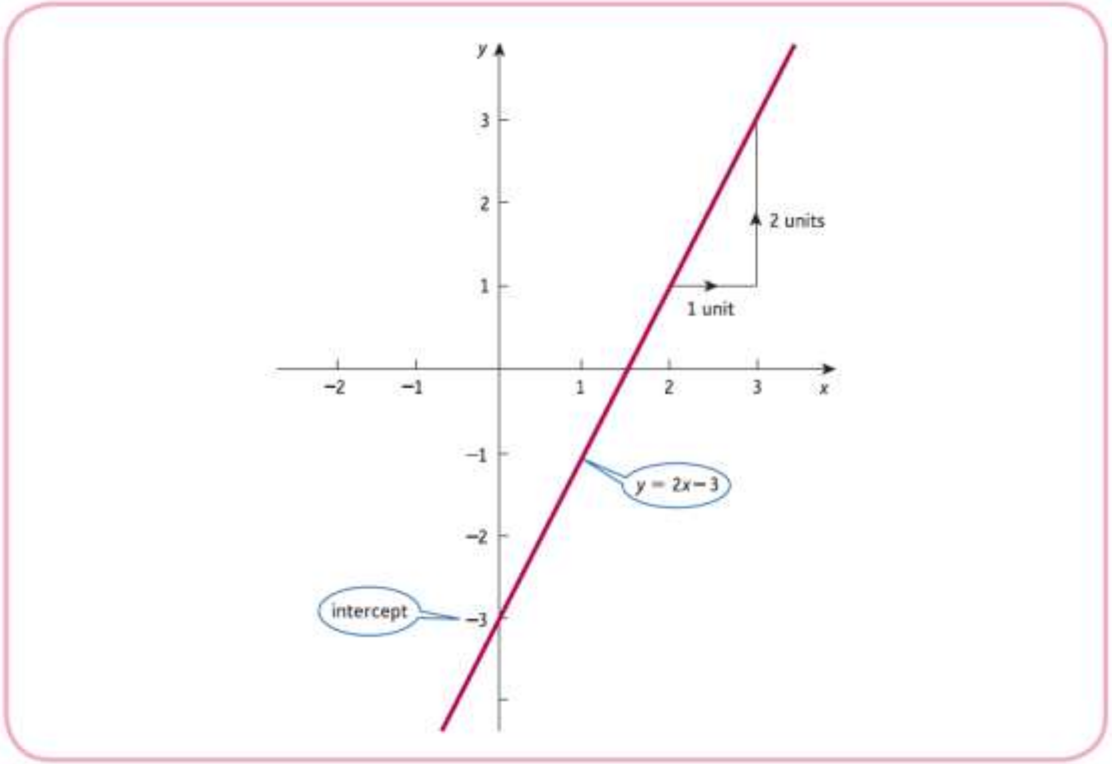

Figure 1.6

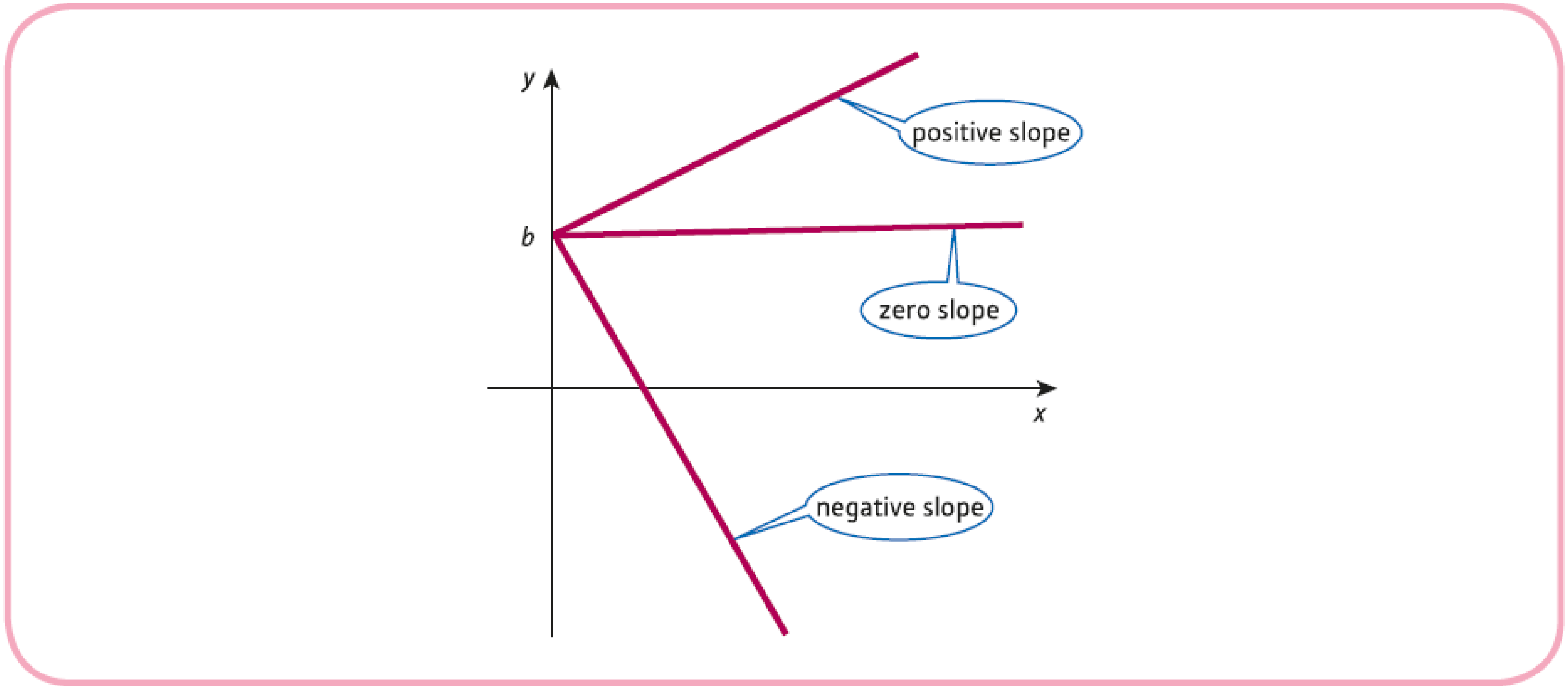

Figure 1.7

corresponding decrease in y, indicating that the line is downhill. If a is zero, then the equation is just

$$y = b$$

indicating that y is fixed at b and the line is horizontal. The three cases are illustrated in Figure 1.7.

It is important to appreciate that in order to use the slope–intercept approach, it is necessary for the equation to be written as

$$y = ax + b$$

If a linear equation does not have this form, it is usually possible to perform a preliminary rearrangement to isolate the variable y on the left-hand side.

For example, to use the slope–intercept approach to sketch the line

$$2x + 3y = 12$$

we begin by removing the x term from the left-hand side. Subtracting $2x$ from both sides gives

$$3y = 12 - 2x$$

and dividing both sides by 3 gives

$$y = 4 - \tfrac{2}{3}x$$

This is now in the required form with $a = -2/3$ and $b = 4$. The line is sketched in Figure 1.8. A slope of $-2/3$ means that, for every 1 unit along, we go 2/3 units down (or, equivalently, for every 3 units along, we go 2 units down). An intercept of 4 means that it passes through (0, 4).

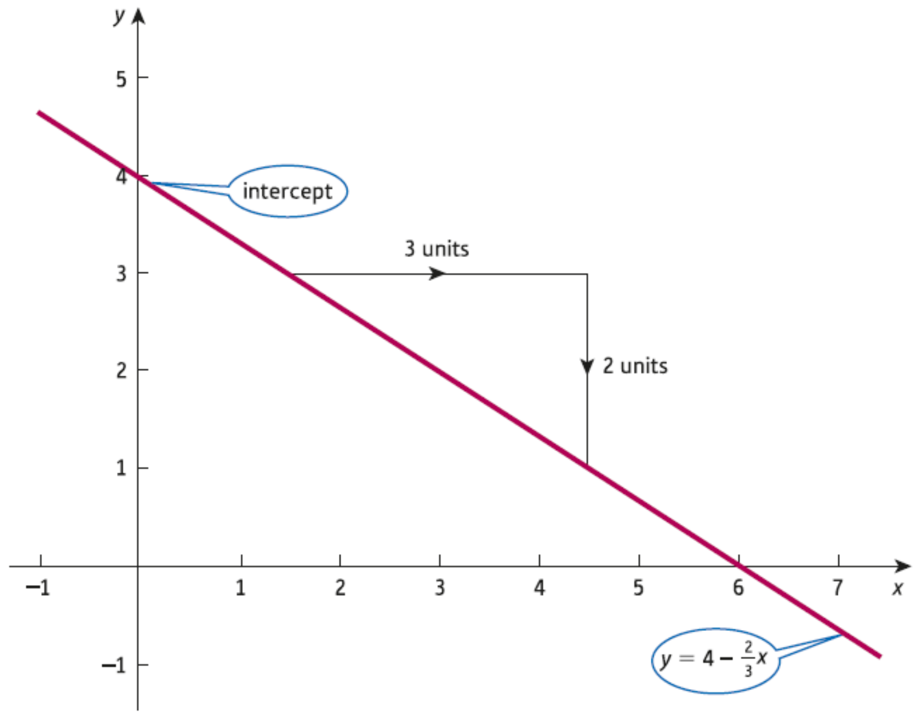

Figure 1.8

Practice Problem

6. Use the slope–intercept approach to sketch the lines

(a) $y = x + 2$

(b) $4x + 2y = 1$

Key Terms

Coefficient A numerical multiplier of the variables in an algebraic term, such as the numbers 4 and 7 in the expression $4x + 7yz^2$.

Coordinates A set of numbers that determine the position of a point relative to a set of axes.

Intercept The point(s) where a graph crosses one of the coordinate axes.

Linear equation An equation of the form $dx + ey = f$.

Origin The point where the coordinate axes intersect.

Simultaneous linear equations A set of linear equations in which there are (usually) the same number of equations and unknowns. The solution consists of values of the unknowns which satisfy all of the equations at the same time.

Slope of a line Also known as the gradient, it is the change in the value of y when x increases by 1 unit.

x axis The horizontal coordinate axis pointing from left to right.

y axis The vertical coordinate axis pointing upwards.

Exercises 1.3

1, 4, 5, 6(a, c), 7(a, d), 8.

Exercises 1.3*

1.